On the asymptotic stability of bound states in 2D cubic Schrödinger equation

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Abstract

We consider the cubic nonlinear Schrödinger equation in two space dimensions with an attractive potential. We study the asymptotic stability of the nonlinear bound states, i.e. periodic in time localized in space solutions. Our result shows that all solutions with small, localized in space initial data, converge to the set of bound states. Therefore, the center manifold in this problem is a global attractor. The proof hinges on dispersive estimates that we obtain for the non-autonomous, non-Hamiltonian, linearized dynamics around the bound states.

1 Introduction

In this paper we study the long time behavior of solutions of the cubic nonlinear Schrödinger equation (NLS) with potential in two space dimensions (2-d):

$$i\partial_t u(t,x) = [-\Delta_x + V(x)]u + \gamma|u|^2u, \qquad t>0,\ x\in\mathbb{R}^2 \tag{1}$$

$$u(0,x) = u_0(x) \tag{2}$$

where $\gamma \in \mathbb{R}-\{0\}$. The equation has important applications in statistical physics, optics and water waves. It describes certain limiting behavior of Bose-Einstein condensates [8, 14] and propagation of time harmonic waves in wave guides [12, 15, 17]. In the latter, t plays the role of the coordinate along the axis of symmetry of the wave guide.

It is well known that this nonlinear equation admits periodic in time, localized in space solutions (bound states or solitary waves). They can be obtained via both variational techniques [1, 28, 21] and bifurcation methods [19, 21], see also next section. Moreover the set of periodic solutions can be organized as a manifold (center manifold). Orbital stability of solitary waves, i.e. stability modulo the group of symmetries $u \mapsto e^{-i\theta}u$, was first proved in [21, 33], see also [9, 10, 24].

In this paper we are going to show that the center manifold is in fact a global attractor for all small, localized in space initial data. This means that the solution decomposes into a

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modulation of periodic solutions (motion on the center manifold) and a part that decays in time via a dispersion mechanism (radiative part). For a precise statement of hypotheses and the result see Section 3.

Asymptotic stability studies of solitary waves were initiated in the work of A. Soffer and M. I. Weinstein [25, 26], see also [2, 3, 4, 7, 11]. Center manifold analysis was introduced in [19], see also [32]. The techniques developed in these papers do not apply to our problem. Indeed the weaker $L^1 \to L^\infty$ dispersion estimates for Schrödinger operators in 2-d, see (24), compared to 3-d and higher, respectively lack of end point Strichartz estimates in $d = 2$, prevent the bootstrapping argument in [7, 19, 25, 26], respectively [11], from closing. The technique of virial theorem, used in [2, 3, 4] to compensate for the weak dispersion in 1-d, would require at least a quintic nonlinearity in our 2-d case. Finally, in [32], the nonlinearity is localized in space, a feature not present in our case, which allows the author to completely avoid any $L^1 \to L^\infty$ estimates.

To overcome this difficulties we used Strichartz estimates, fixed point and interpolation techniques to carefully analyze the full, time dependent, non-Hamiltonian, linearized dynamics around solitary waves. We obtained dispersive estimates that are similar with the ones for the time independent, Hamiltonian Schrödinger operator, see section 4. Related results have been proved for the 1-d and 3-d case in [20, 13, 23] but their argument does not extend to the 2-d case. We relied on these estimates to understand the nonlinear dynamics via perturbation techniques. We think that our estimates are also useful in approaching the dynamics around large 2-d solitary waves while the techniques that we develop may be used in lowering the power of nonlinearity needed for the asymptotic stability results in 1 and 3-d mentioned in the previous paragraph.

Note that, in 3-d, the case of a center manifold formed by two distinct branches (ground state and excited state) has been analyzed. Under the assumption that the excited branch is sufficiently far away from the ground state one, in a series of papers [27, 29, 30, 31], the authors show asymptotic stability of the ground states with the exception of a finite dimensional manifold where the solution converges to excited states. We cannot extend such a result to our 2-d problem as of now. The reason is the slow convergence in time towards the center manifold, t^{-1+} in 2-d compared to $t^{-3/2}$ in 3-d. This prevents us from even analyzing the projected dynamics on a single branch center manifold, i.e. the evolution of one complex parameter describing the projection of the solution on the center manifold, and obtain, for example, convergence to a periodic orbit as in [4, 19, 26]. However the evolution of this parameter, respectively two parameters in the presence of the excited branch, is given by an ordinary differential equation (ODE), respectively a system of two ODE's, and the contribution of most of the terms can be determined from our estimates, see the discussion in Section 5. We think it is only a matter of time until the remaining ones will be understood.

The paper is organized as follows. In the next section we discuss previous results regarding the manifold of periodic solutions that we subsequently need. In section 3 we formulate and prove our main result. As we mentioned before the proof relies on certain estimates for the linear dynamics which we prove in section 4. We conclude with possible extensions and comments in section 5.

Notations: $H = -\Delta + V$;

$L^p = \{f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f$ measurable and $\int_{\mathbb{R}^2} |f(x)|^p dx < \infty\}$, $\|f\|_p = \left(\int_{\mathbb{R}^2} |f(x)|^p dx\right)^{1/p}$

denotes the standard norm in these spaces;

$< x >= (1 + |x|^2)^{1/2}$, and for $\sigma \in \mathbb{R}$, L^2_σ denotes the L^2 space with weight $< x >^{2\sigma}$, i.e. the space of functions $f(x)$ such that $< x >^\sigma f(x)$ are square integrable endowed with the norm $\|f(x)\|_{L^2_\sigma} = \| < x >^\sigma f(x)\|_2$;

$\langle f, g\rangle = \int_{\mathbb{R}^2} \overline{f}(x)g(x)dx$ is the scalar product in L^2 where $\overline{f}$ = the complex conjugate of the complex number f;

P_c is the projection on the continuous spectrum of H in L^2;

H^n denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order n in L^2, $\|\cdot\|_{H^n}$ denotes the standard norm in this spaces.

2 Preliminaries. The center manifold.

The center manifold is formed by the collection of periodic solutions for (1):

$$u_E(t, x) = e^{-iEt}\psi_E(x) \tag{3}$$

where $E \in \mathbb{R}$ and $0 \not\equiv \psi_E \in H^2(\mathbb{R}^2)$ satisfy the time independent equation:

$$[-\Delta + V]\psi_E + \gamma|\psi_E|^2\psi_E = E\psi_E \tag{4}$$

Clearly the function constantly equal to zero is a solution of (4) but (iii) in the following hypotheses on the potential V allows for a bifurcation with a nontrivial, one parameter family of solutions:

(H1) Assume that

(i) There exists $C > 0$ and $\rho > 3$ such that:

$$|V(x)| \leq C < x >^{-\rho}, \quad \text{for all } x \in \mathbb{R}^2;$$

(ii) 0 is a regular point[1] of the spectrum of the linear operator $H = -\Delta + V$ acting on L^2;

(iii) H acting on L^2 has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector ψ_0. It is well known that $\psi_0(x)$ can be chosen strictly positive and exponentially decaying as $|x| \to \infty$.

Conditions (i)-(ii) guarantee the applicability of dispersive estimates of Murata [16] and Schlag [22] to the Schrödinger group e^{-iHt}, see section 4. In particular (i) implies the local well posedness in H^1 of the initial value problem (1-2), see section 3.

Condition (iii) guarantees bifurcation of nontrivial solutions of (4) from (E_0, ψ_0). In Section 5, we discuss the possible effects of relaxing (iii) to allow for finitely many negative eigenvalues. We construct the center manifold by applying the standard bifurcation argument in Banach spaces [18] for (4) at $E = E_0$. We follow [19] and decompose the solution of (4) in its projection onto the discrete and continuous part of the spectrum of H :

$$\psi_E = a\psi_0 + h, \quad a = \langle\psi_0, \psi_E\rangle, \ h = P_c\psi_E.$$

[1] see [22, Definition 7] or $M_\mu = \{0\}$ in relation (3.1) in [16]

Using the notations

$$f_p(a,h) \equiv \langle \psi_0, |a\psi_0 + h|^2(a\psi_0 + h)\rangle, \tag{5}$$
$$f_c(a,h) \equiv P_c|a\psi_0 + h|^2(a\psi_0 + h), \tag{6}$$

and projecting (4) onto ψ_0 and its orthogonal complement = Range P_c we get:

$$h = -\gamma(H-E)^{-1} f_c(a,h) \tag{7}$$
$$E_0 - E = -\gamma a^{-1} f_p(a,h). \tag{8}$$

Although we are using milder hypothesis on V the argument in the Appendix of [19] can be easily adapted to show that:

$$\mathcal{F}(E,a,h) = h + \gamma(H-E)^{-1} f_c(a,h)$$

is a C^1 function from $(-\infty,0)\times\mathbb{C}\times L^2_\sigma\cap H^2$ to $L^2_\sigma\cap H^2$ and $\mathcal{F}(E_0,0,0) = 0$, $D_h\mathcal{F}(E_0,0,0) = I$. Therefore the implicit function theorem applies to equation (7) and leads to the existence of $\delta_1 > 0$ and the C^1 function $\tilde{h}(E,a)$ from $(E_0-\delta_1, E_0+\delta_1)\times\{a\in\mathbb{C} \ : \ |a| < \delta_1\}$ to $L^2_\sigma \cap H^2$ such that (7) has a unique solution $h = \tilde{h}(E,a)$ for all $E \in (E_0-\delta_1, E_0+\delta_1)$ and $|a| < \delta_1$. Note that if (a,h) solves (7) then $(e^{i\theta}a, e^{i\theta}h)$, $\theta\in[0,2\pi)$ is also a solution, hence by uniqueness we have:

$$\tilde{h}(E,a) = \frac{a}{|a|}\tilde{h}(E,|a|). \tag{9}$$

Because ψ_0 is real valued, we could apply the implicit function theorem to (7) under the restriction $a\in\mathbb{R}$ and h in the subspace of real valued functions as it is actually done in [19]. By uniqueness of the solution we deduce that $\tilde{h}(E,|a|)$ is a real valued function.

Replacing now $h = \tilde{h}(E,a)$ in (8) and using (5) and (9) we get the equivalent formulation:

$$E_0 - E = -\gamma|a|^{-1} f_p(|a|, \tilde{h}(E,|a|)). \tag{10}$$

To this we can apply again the implicit function theorem by observing that $G(E,a) = E_0 - E + \gamma|a|^{-1}f_p(|a|,\tilde{h}(E,|a|))$ is a C^1 function [19, Appendix] from $(E_0-\delta_1, E_0+\delta_1)\times(-\delta_1,\delta_1)$ to $\mathbb{R}$ with the properties $G(E_0,0) = 0$, $\partial_E G(E_0,0) = -1$. We obtain the existence of $0 < \delta \le \delta_1$, $0 < \delta_E \le \delta_1$ and the C^1 function $\tilde{E} : (-\delta,\delta) \mapsto (E_0-\delta_E, E_0+\delta_E)$ such that, for $|E-E_0| < \delta_E$, $|a| < \delta$, the unique solution of (8) with $h = \tilde{h}(E,a)$, is given by $E = \tilde{E}(|a|)$. If we now define:

$$h(a) \equiv \frac{a}{|a|}\tilde{h}(E(|a|,|a|)$$

we have the following center manifold result:

Proposition 2.1 *There exist $\delta_E, \delta > 0$ and the C^1 function*

$$h : \{a\in\mathbb{C} \ : \ |a| < \delta\} \mapsto L^2_\sigma \cap H^2,$$

such that for $|E-E_0| < \delta_E$ the eigenvalue problem (4) has a unique solution up to multiplication with $e^{i\theta}$, $\theta\in[0,2\pi)$, which can be represented as:

$$\psi_E = a\psi_0 + h(a), \qquad \langle\psi_0, h(a)\rangle = 0, \ |a| < \delta.$$

Since $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$ the proposition implies that $\psi_E \in L^2_\sigma$. A regularity argument, see [25], gives a stronger result:

Corollary 2.1 *For any $\sigma \in \mathbb{R}$, there exists a finite constant C_σ such that:*

$$\| < x >^\sigma \psi_E\|_{H^2} \leq C_\sigma \|\psi_E\|_{H^2}.$$

We are now ready to prove our main result.

3 Main Result. The collapse on the center manifold.

Theorem 3.1 *Assume that hypothesis (H1) is valid and fix $\sigma > 2$. Then there exists an $\varepsilon_0 > 0$ such that for all initial conditions $u_0(x)$ satisfying*

$$\max\{\|u_0\|_{L^2_\sigma}, \|u_0\|_{H^1}\} \leq \varepsilon_0$$

the initial value problem (1)-(2) is globally well-posed in H^1.

Moreover, for all $t \in \mathbb{R}$ and $p > 2$, we have that:

$$u(t,x) = \underbrace{a(t)\psi_0(x) + h(a(t))}_{\psi_E(t)} + r(t,x) \tag{11}$$

$$\|r(t)\|_{L^2_{-\sigma}} \leq \frac{\bar{C}_1 \varepsilon_0}{(1+|t|)^{1-2/p}}$$

$$\|r(t)\|_{L^p} \leq \frac{\bar{C}_2 \varepsilon_0 \log(2+|t|)}{(1+|t|)^{1-2/p}}$$

with the constants $\bar{C}_1, \bar{C}_2$,independent of ε_0 and $\bar{C}_2$ depending on $p > 2$.

Before proving the theorem let us note that (11) decomposes the evolution of the solution of (1)-(2) into an evolution on a center manifold $\psi_E(t)$ and the "distance" from the center manifold $r(t)$. The estimates on the latter show collapse of solution onto the center manifold. The evolution on the center manifold is determined by equation (12) below. We discuss it in Section 5.

Proof of Theorem 3.1. It is well known that under hypothesis (H1)(i) the initial value problem (1)-(2) is locally well posed in the energy space H^1 and its L^2 norm is conserved, see for example [5, Corollary 4.3.3. at p. 92]. Global well posedness follows via energy estimates from $\|u_0\|_2$ small, see [5, Remark 6.1.3 at p. 165].

In particular we can define

$$a(t) = \langle \psi_0, u(t) \rangle, \quad \text{for } t \in \mathbb{R}.$$

Cauchy-Schwarz inequality implies

$$|a(t)| \leq \|u(t)\|_2 \|\psi_0\|_2 = \|u_0\|_2 \leq \varepsilon_0, \text{ for } t \in \mathbb{R}$$

where we also used conservation of L^2 norm of u. Hence, if we choose $\varepsilon_0 < \delta$ we can define $h(a(t)),\ t \in \mathbb{R}$, see Proposition 2.1. We then obtain (11) where

$$r(t) = u(t) - a(t)\psi_0 - h(a(t)), \quad \langle \psi_0, r(t)\rangle \equiv 0.$$

The solution is now described by the scalar $a(t) \in \mathbb{C}$ and $r(t) \in C(\mathbb{R}, H^1)$. Their equations are obtained by projecting (1) onto ψ_0 and its orthogonal complement in L^2 :

$$i\frac{da}{dt} = E(|a(t)|)a(t) + \gamma\langle \psi_0, 2|\psi_E|^2 r + \psi_E^2\bar{r} + 2\psi_E|r|^2 + \bar{\psi}_E r^2 + |r|^2 r\rangle \tag{12}$$

$$\begin{aligned} i\frac{\partial r}{\partial t} &= Hr + \gamma P_c[2|\psi_E|^2 r + \psi_E^2\bar{r} + 2\psi_E|r|^2 + \bar{\psi}_E r^2 + |r|^2 r] \\ &\quad -\gamma Dh|_{a(t)}\langle \psi_0, 2|\psi_E|^2 r + \psi_E^2\bar{r} + 2\psi_E|r|^2 + \bar{\psi}_E r^2 + |r|^2 r\rangle \end{aligned} \tag{13}$$

where we used the identities $\psi_E = a\psi_0 + h(a)$ and $Dh|_a[a] = E(|a|)h(a)$ for $a \in \mathbb{C},\ |a| < \delta$.

In order to obtain the estimates for $r(t)$, we analyze equation (13). In the next section we study its linear part:

$$\begin{cases} i\frac{\partial z}{\partial t} & = Hz + \gamma P_c[2|\psi_E|^2 z + \psi_E^2\bar{z} - \gamma Dh|_{a(t)}\langle \psi_0, 2|\psi_E|^2 z + \psi_E^2\bar{z}\rangle] \\ z(s) & = v \end{cases}$$

Let us denote by $\Omega(t,s)v$ the operator which associates to the function v the solution of the above equation:

$$\Omega(t,s)v \stackrel{\text{def}}{=} z \tag{14}$$

The estimates that we need for this linear propagator are proved in the next section.

Now, using Duhamel's principle (13) becomes

$$\begin{aligned} r(t) = \Omega(t,0)r(0) + \int_0^t \Omega(t,s)\{&\gamma P_c[2\psi_E|r(s)|^2 + \bar{\psi}_E r^2(s) + |r|^2 r(s)] \\ &-\gamma Dh|_{a(t)}[\langle \psi_0, 2\psi_E|r|^2 + \bar{\psi}_E r^2 + |r|^2 r\rangle]\}ds \end{aligned} \tag{15}$$

It is here where we differ essentially from the approach for the 1-d case [2, 3, 4] and 3-d case [7, 25, 26, 19]. The right hand side of our equation contains only nonlinear terms in r. Hence, if we make the ansatz $r(t) \sim (1+t)^{-3/4}$ then the quadratic and cubic terms in (15) decay like $(1+s)^{-6/4}$ respectively $(1+s)^{-9/4}$. Both are integrable functions in time, hence, via convolution estimates, the integral term on the right hand side decays like $\Omega(t,0)$. We have a chance of "closing" the ansatz provided $\Omega(t,0) \sim (1+t)^{-3/4}$. Contrast this with the case in the above cited papers where a linear term in r is present on the right hand side. Same argument leads to a loss of 1/4 power decay in the linear term and requires $\Omega(t,0) \equiv e^{-iHt} \sim (1+t)^{-1-\delta},\ \delta > 0$. for closing. This turns out to be impossible in L^p norms in 2-d, see (24), while the use of weighted L^2 norms, see (23), for delocalized terms as the cubic term in (15), would require compensation via virial inequalities, see [2], which needs a much higher power nonlinearity than cubic in the delocalized terms[2].

The following Lemma makes the above heuristic argument rigorous:

[2]Heuristically we arrived at quintic power nonlinearity, hence this technique may be applicable to the quintic Schrödinger in 2-D but definitely not to the cubic one.

Lemma 3.1 *There exists $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that if $\| < x >^{\sigma} \psi_E\|_{H^2} \leq \varepsilon_1$ and the initial condition $r(0)$ satisfies*

$$\max\{\|r(0)\|_{L^2_\sigma}, \|r(0)\|_{H^1}\} < \varepsilon_2$$

the initial value problem (15) is globally well-posed in $C(\mathbb{R}, L^2_{-\sigma} \cap L^2 \cap L^p)$, $6 \leq p < \infty$ and for all $t \in \mathbb{R}$:

$$\begin{aligned} \|r(t)\|_{L^2_{-\sigma}} &\leq \frac{\bar{C}_1 \varepsilon_0}{(1+|t|)^{1-2/p}} \\ \|r(t)\|_{L^p} &\leq \frac{\bar{C}_2 \varepsilon_0 \log(2+|t|)}{(1+|t|)^{1-2/p}} \\ \|r(t)\|_{L^2} &\leq \bar{C}_3 \varepsilon_0 \end{aligned}$$

with $\bar{C}_1, \bar{C}_2, \bar{C}_3$ independent of ε_0 and $\bar{C}_2$ depending on $p \geq 6$.

Note that the Lemma finishes the proof of the theorem. Indeed, we now have two solutions of (13), one in $C(\mathbb{R}, H^1)$ from classical well posedness theory and one in $C(\mathbb{R}, L^2_{-\sigma} \cap L^2 \cap L^p)$, $p \geq 6$ from the Lemma. Using uniqueness and the continuous embedding of H^1 in $L^2_{-\sigma} \cap L^2 \cap L^p$, we infer that the two solutions must coincide. Therefore, the time decaying estimates in the Lemma hold also for the H^1 solution. The L^p, $2 < p < 6$ estimates in the theorem follow from interpolation:

$$\|r(t)\|_{L^p} \leq \|r(t)\|_{L^2}^{3/p-1/2} \|r(t)\|_{L^6}^{3/2-3/p} \leq \frac{\bar{C}_3 \bar{C}_2 \varepsilon_0 \log(2+|t|)}{(1+|t|)^{1-2/p}}$$

It remains to prove the Lemma:

Proof of Lemma 3.1.

Fix $p \geq 6$. We will show that (15) has a solution by applying the contraction principle in the functional space

$$\begin{aligned} Y = \{f : R \to L^2_{-\sigma} \cap L^p \cap L^2 | \ &\sup_{t \geq 0} (1+|t|)^{1-\frac{2}{p}} \|f(t)\|_{L^2_{-\sigma}} < \infty \\ &\sup_{t \geq 0} \frac{(1+|t|)^{1-2/p}}{\log(2+|t|)} \|f(t)\|_{L^p} < \infty, \sup_{t \geq 0} \|f(t)\|_{L^2} < \infty\} \end{aligned}$$

endowed with the norm

$$\|f\|_Y = \max\{\sup_{t \geq 0} (1+|t|)^{1-\frac{2}{p}} \|f(t)\|_{L^2_{-\sigma}}, \sup_{t \geq 0} \frac{(1+|t|)^{1-2/p}}{\log(2+|t|)} \|f(t)\|_{L^p}, \sup_{t \geq 0} \|f(t)\|_{L^2}\}$$

To this extent we consider the operator N defined on functions in Y as

$$\begin{aligned} (Nu)(t) = \Omega(t,s)v + \int_0^t \Omega(t,\tau)\gamma\{&P_c[2\psi_E|u|^2 + \bar{\psi}_E u^2 + |u|^2 u] \\ &-Dh|_{a(\tau)}\langle \psi_0, 2\psi_E|u|^2 + \bar{\psi}_E u^2 + |u|^2 u\rangle\} d\tau \end{aligned}$$

We will need some properties of the operator N which are summarized in

Lemma 3.2 *We have*
(i) The range of N is Y i.e. $N : Y \to Y$ is well defined.
(ii) There exists $\tilde{C} > 0$ such that

$$\|Nu_1 - Nu_2\|_Y \leq \tilde{C}(\|u_1\|_Y + \|u_2\|_Y + \|u_1\|_Y^2 + \|u_2\|_Y^2)\|u_1 - u_2\|_Y$$

In particular N is locally Lipschitz.

Moreover, $\tilde{C} = \tilde{C}(C, C_p, C_{p,q_0'})$ where the constants $C, C_p, C_{p,q_0'}$ are those from the linear estimates for $\Omega(t,s)$ (see Theorems 4.1, 4.2 in the next section).

Proof of Lemma 3.2 Let us observe that it will suffice to show part *(ii)* and then using the fact that $N(0) \equiv 0$ we will have part *(i)*. Indeed, part *(ii)* will give us that for $u_1, u_2 \in Y$ we have $Nu_1 - Nu_2 \in Y$. Taking $u_2 \equiv 0$ and since $N(0) \equiv 0$ this will imply that $Nu_1 \in Y$.

Thus, take $u_1, u_2 \in Y$ and consider the difference $Nu_1 - Nu_2$, which is

$$\begin{aligned}(Nu_1 - Nu_2)(t) = \int_0^t \Omega(t,\tau)\gamma\{P_c[&2\psi_E(|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E(u_1 - u_2)(u_1 + u_2) \\ &+(u_1 - u_2)|u_1|^2 + (|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)] \\ -Dh|_{a(\tau)}\langle\psi_0, &2\psi_E(|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E(u_1 - u_2)(u_1 + u_2) + \\ &(u_1 - u_2)|u_1|^2 + (|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)\rangle\}d\tau\end{aligned}$$

The $L^2_{-\sigma}$ estimate We can work under the less restrictive hypothesis: $4 < p < \infty$. Let $L^{p'}$ be the dual of L^p, i.e. $1/p' + 1/p = 1$. We have

$$\begin{aligned}\|Nu_1 - Nu_2\|_{L^2_{-\sigma}} \leq &\int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^2_{-\sigma}} \times \\ &\times \| \underbrace{2\psi_E < x >^\sigma (|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E < x >^\sigma (u_1 - u_2)(u_1 + u_2)}_{A} \|_{L^2} d\tau \\ &+ \int_0^t \|\Omega(t,\tau)\|_{L^{p'} \to L^2_{-\sigma}} \times \\ &\times \| \underbrace{(u_1 - u_2)|u_1|^2}_{B_1} + \underbrace{(|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)}_{B_2} \|_{L^{p'}} d\tau \\ &+ \int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^2_{-\sigma}} \|Dh|_{a(\tau)}\|_{L^2_\sigma} \times \\ &\times \{ \underbrace{|\langle\psi_0, 2\psi_E(|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E(u_1 - u_2)(u_1 + u_2) >|}_{F} + \\ &+ \underbrace{| < \psi_0, (u_1 - u_2)|u_1|^2 + (|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)\rangle|}_{G} \} d\tau\end{aligned} \tag{16}$$

To estimate the term A we observe that

$$\| < x >^\sigma \psi_E(|u_1| - |u_2|)(|u_1| + |u_2|)\|_{L^2} \leq \| < x >^\sigma \psi_E\|_{L^\alpha} \|u_1 - u_2\|_{L^p} \||u_1| + |u_2|\|_{L^p} \tag{17}$$

with $\frac{1}{\alpha} + \frac{2}{p} = \frac{1}{2}$. Then

$$\int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^2_{-\sigma}} A(\tau) d\tau$$

$$\leq \int_0^t \frac{C}{(1+|t-\tau|)\log^2(2+|t-\tau|)} \cdot 3\|\psi_E < x >^\sigma |u_1 - u_2|(|u_1| + |u_2|)\|_{L^2} d\tau$$

$$\leq 3C\tilde{C}_1 \int_0^t \frac{\log^2(2+|\tau|)}{(1+|t-\tau|)\log^2(2+|t-\tau|)} \frac{\||u_1| - |u_2|\|_Y}{(1+|\tau|)^{(1-\frac{2}{p})}} \cdot \frac{\||u_1| + |u_2|\|_Y}{(1+|\tau|)^{(1-\frac{2}{p})}}$$

$$\leq 3C\tilde{C}_1\tilde{C}_2(\|u_1\|_Y + \|u_2\|_Y) \frac{\|u_1 - u_2\|_Y}{(1+|t|)\log^2(2+|t|)}$$

where for the first inequality we used *Theorem* 4.1,part *(i)*. The constants are given by $\tilde{C}_1 = \sup_{t>0} \| < x >^\sigma \psi_E\|_{L^\alpha}$ and $\tilde{C}_2 = \sup_{t>0}(1+|t|)\log^2(2+|t|)\int_0^t \frac{\log^2(2+|\tau|)}{(1+|t-\tau|)\log^2(2+|t-\tau|)} \cdot \frac{d\tau}{(1+|\tau|)^{2-\frac{4}{p}}} < \infty$, because $p > 4$.

To estimate the cubic terms B_1, B_2 we can not use the term ψ_E as before, and this is what forces us to work in the L^p space. We have:

$$\|(u_1 - u_2)|u_1|^2\|_{L^{p'}} \leq \|u_1 - u_2\|_{L^p}\|u_1\|^2_{L^\alpha}$$

respectively

$$\|(u_1 - u_2)(u_2|u_1| + u_2|u_2|)\|_{L^{p'}} \leq \|u_1 - u_2\|_{L^p}\|u_2\|_{L^\alpha}(\|u_1\|_{L^\alpha} + \|u_2\|_{L^\alpha})$$

with $\frac{2}{\alpha} + \frac{1}{p} = \frac{1}{p'}$. Since $4 \leq p$ we have $2 \leq \alpha \leq p$. Therefore we can again interpolate:

$$\|u_i\|_{L^\alpha} \leq \|u_i\|^{1-b}_{L^2}\|u_i\|^b_{L^p}, i = 1, 2,$$

where $\frac{1}{\alpha} = \frac{1-b}{2} + \frac{b}{p}$. Combining these relations we obtain for B_1 :

$$\|(u_1 - u_2)|u_1|^2\|_{L^{p'}} \leq \|u_1 - u_2\|_{L^p}\|u_1\|^{2(1-b)}_{L^2}\|u_1\|^{2b}_{L^p} \tag{18}$$

respectively, for B_2 :

$$\|(u_1 - u_2)(u_2|u_1| + u_2|u_2|)\|_{L^{p'}} \leq \|u_1 - u_2\|_{L^p}\|u_2\|^{1-b}_{L^2}\|u_2\|^b_{L^p}(\|u_1\|^{1-b}_{L^2}\|u_1\|^b_{L^p} + \|u_2\|^{1-b}_{L^2}\|u_2\|^b_{L^p}) \tag{19}$$

with

$$\frac{2(1-b)}{2} + \frac{2b}{p} + \frac{1}{p} = \frac{1}{p'}.$$

A consequence of this relation and of $p < \infty$ is:

$$(1 - \frac{2}{p})(1+2b) = 1 + 2/p > 1 \tag{20}$$

which will play an essential role in what follows.

Thus, the estimate for the term containing $B_1 + B_2$ is

$$\int_0^t \|\Omega(t,\tau)P_c\|_{L^{p'}\to L^2_{-\sigma}}\|B_1+B_2\|_{L^{p'}} \leq C_p(\|u_1\|_Y^2+\|u_2\|_Y^2)\|u_1-u_2\|_Y \times$$

$$\times \int_0^t \frac{\log(2+|\tau|)^{(1+2b)}}{|t-\tau|^{1-\frac{2}{p}}} \cdot \frac{1}{(1+|\tau|)^{(1-\frac{2}{p})(1+2b)}} d\tau$$

$$\leq C_p\tilde{C}_3(\|u_1\|_Y^2+\|u_2\|_Y^2)\frac{\|u_1-u_2\|_Y}{(1+|t|)^{1-2/p}}$$

where for the first inequality we used *Theorem* 4.1, part *(ii)*, inequalities (18), (19) and the definition of the norm in Y. For the last inequality we used the fact that $(1-\frac{2}{p})(1+2b) > 1$ (see (20)) with $\tilde{C}_3 = \sup_{t>0}(1+|t|)^{1-2/p}\int_0^t \frac{\log(2+|\tau|)^{(1+2b)}}{|t-\tau|^{1-\frac{2}{p}}}\frac{1}{(1+|\tau|)^{(1-\frac{2}{p})(1+2b)}}d\tau < \infty$.

For estimating the term containing F we have

$$|F| \leq 3\|\psi_0\|_{L^\infty}\|\psi_E\|_{L^\alpha}\|u_1-u_2\|_{L^p}(\|u_1\|_{L^p}+\|u_2\|_{L^p})$$

$\frac{1}{\alpha}+\frac{2}{p} = 1$. Then, the term containing F is estimated as the term containing A with $\tilde{C}_1$ replaced by $\tilde{C}_4 = \sup_{t>0}\|Dh|_{a(\tau)}\|_{L^2_\sigma}\|\psi_0\|_{L^\infty}\|\psi_E\|_{L^\alpha}$.

We estimate G as

$$|G| \leq \|\psi_0 < x >^\sigma\|_{L^\alpha}\|u_1-u_2\|_{L^2_{-\sigma}}(\|u_1\|_{L^p}^2+\|u_2\|_{L^p}^2+\|u_1\|_{L^p}\|u_2\|_{L^p})$$

with $\frac{1}{\alpha}+\frac{1}{2}+\frac{2}{p} = 1$. Then the term containing G is estimated as

$$\int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma\to L^2_{-\sigma}}|G|d\tau \leq 3\int_0^t \frac{C\|\psi_0 < x >^\sigma\|_{L^\alpha}}{(1+|t-\tau|)\log^2(2+|t-\tau|)}\cdot\frac{\log^2(2+|\tau|)}{(1+|\tau|)^{3-6/p}}d\tau$$

$$\leq 3C\tilde{C}_5(\|u_1\|_Y^2+\|u_2\|_Y^2)\frac{\|u_1-u_2\|_Y}{(1+|t|)\log^2(2+|t|)}$$

with $\tilde{C}_5 = \|\psi_0 < x >^\sigma\|_{L^\alpha}\sup_{t>0}(1+|t|)\log^2(2+|t|)\int_0^t \frac{\log^2(2+|\tau|)}{(1+|t-\tau|)\log^2(2+|t-\tau|)(1+|\tau|)^{3-6/p}}d\tau < \infty$ because $p > 3$.

The L^p estimate: With $p',\ q_0',\ q'$ given by Theorem 4.2, we have

$$\|Nu_1 - Nu_2\|_{L^p} \leq \int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^p} \times$$
$$\times \| \underbrace{2\psi_E < x >^\sigma (|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E < x >^\sigma (u_1 - u_2)(u_1 + u_2)}_{A} \|_{L^2} d\tau$$
$$+ \int_0^t \|\Omega(t,\tau)\|_{L^{p'} \cap L^{q_0'} \cap L^{q'} \to L^p} \times$$
$$\times \| \underbrace{(u_1 - u_2)|u_1|^2}_{B_1} + \underbrace{(|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)}_{B_2} \|_{L^{p'} \cap L^{q_0'} \cap L^{q'}} d\tau$$
$$+ \int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^p} \|Dh|_{a(\tau)}\|_{L^2_\sigma} \times$$
$$\times \{ \underbrace{|\langle \psi_0, 2\psi_E(|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E(u_1 - u_2)(u_1 + u_2)\rangle|}_{F} +$$
$$+ \underbrace{|\langle \psi_0, (u_1 - u_2)|u_1|^2 + (|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)\rangle|}_{G} \} d\tau \qquad (21)$$

The term A can be treated exactly as before and for the Ω term we use *Theorem* 4.1 part *(iii)*. Since $1 < p',\ q_0',\ q' \leq 4/3$, we can estimate the B_1, B_2 terms in each of the norms $L^{p'},\ L^{q_0},\ L^{q'}$ as we did above for their $L^{p'}$ norm only. For Ω we use *Theorem* 4.2, part *(iii)*. The terms F and G are also treated as in the previous case. The convolution integrals in (21) will all decay like $(1+|t|)^{-(1-2/p)}$ except the second one which will have a logarithmic correction dominated by $\log(2+|t|)$.

The L^2 estimate: We have

$$\|Nu_1 - Nu_2\|_{L^2} \leq \int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^2} \times$$
$$\times \| \underbrace{2\psi_E < x >^\sigma (|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E < x >^\sigma (u_1 - u_2)(u_1 + u_2)}_{A} \|_{L^2} d\tau$$
$$+ \int_0^t \|\Omega(t,\tau)\|_{L^{p'} \cap L^2 \to L^2} \times$$
$$\times \| \underbrace{(u_1 - u_2)|u_1|^2}_{B_1} + \underbrace{(|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)}_{B_2} \|_{L^{p'} \cap L^2} d\tau$$
$$+ \int_0^t \|\Omega(t,\tau)\|_{L^2_\sigma \to L^2} \|Dh|_{a(\tau)}\|_{L^2_\sigma} \times$$
$$\times \{ \underbrace{|\langle \psi_0, 2\psi_E(|u_1| - |u_2|)(|u_1| + |u_2|) + \bar{\psi}_E(u_1 - u_2)(u_1 + u_2)\rangle|}_{F} +$$
$$+ \underbrace{|\langle \psi_0, (u_1 - u_2)|u_1|^2 + (|u_1| - |u_2|)(u_2|u_1| + u_2|u_2|)\rangle|}_{G} \} d\tau \qquad (22)$$

We estimate the term A as in (17) while the estimates in $L^{p'}$ for B_1, B_2 are as in (18) and (19). For their estimate in L^2 norm we use

$$\|(u_1 - u_2)|u_1|^2\|_{L^2} \leq \|u_1 - u_2\|_{L^p} \|u_1\|^2_{L^\alpha}$$

respectively

$$\|(u_1-u_2)(u_2|u_1|+u_2|u_2|)\|_{L^2} \leq \|u_1-u_2\|_{L^p}\|u_2\|_{L^\alpha}(\|u_1\|_{L^\alpha}+\|u_2\|_{L^\alpha})$$

with $\frac{2}{\alpha}+\frac{1}{p}=\frac{1}{2}$. Since $6 \leq p$ we have $4 \leq \alpha \leq p$. Therefore we can again interpolate:

$$\|u_i\|_{L^\alpha} \leq \|u_i\|_{L^2}^{1-b}\|u_i\|_{L^p}^{b}, i=1,2,$$

where $\frac{1}{\alpha}=\frac{1-b}{2}+\frac{b}{p}$. Combining these relations we obtain for B_1 :

$$\|(u_1-u_2)|u_1|^2\|_{L^2} \leq \|u_1-u_2\|_{L^p}\|u_1\|_{L^2}^{2(1-b)}\|u_1\|_{L^p}^{2b}$$

respectively, for B_2 :

$$\|(u_1-u_2)(u_2|u_1|+u_2|u_2|)\|_{L^2} \leq \|u_1-u_2\|_{L^p}\|u_2\|_{L^2}^{1-b}\|u_2\|_{L^p}^{b}(\|u_1\|_{L^2}^{1-b}\|u_1\|_{L^p}^{b}+\|u_2\|_{L^2}^{1-b}\|u_2\|_{L^p}^{b})$$

with

$$\frac{2(1-b)}{2}+\frac{2b}{p}+\frac{1}{p}=\frac{1}{2}.$$

A consequence of this relation is:

$$(1-\frac{2}{p})(1+2b)=2$$

Using now the definition of the norm in Y we will have:

$$\|B_1+B_2\|_L^2 \leq \|u_1-u_2\|_Y(\|u_1\|_Y^2+\|u\|_2^2)\frac{\log^{2p/(p-2)}(2+|t|)}{(1+|t|)^2}$$

The previous estimates for F and G suffice here as well.

Recalling from *Theorem* 4.1, part *(iii)* and *Theorem* 4.2, part *(ii)*, that $\|\Omega(t,\tau)\|_{L^2_\sigma\to L^2}$ and $\|\Omega(t,\tau)\|_{L^{p'}\cap L^2\to L^2}$ are bounded, and combining with the estimates above, as well as taking into account the definition of the functional space Y we have that

$$\|Nu_1-Nu_2\|_{L^2} \leq C\|u_1-u_2\|_Y[\tilde{C}_6(\|u_1\|_Y+\|u_2\|_Y)+\tilde{C}_7(\|u_1\|_Y^2+\|u_2\|_Y^2)]$$

with $\tilde{C}_6=\sup_{t\geq 0}\int_0^t \frac{\log^2(2+|\tau|)}{(1+|\tau|)^{(2-4/p)}}d\tau<\infty$ and $\tilde{C}_7=\sup_{t\geq 0}\int_0^t \frac{\log^{(1+2b)}(2+|\tau|)}{(1+|\tau|)^{(1+2b)(1-2/p)}}d\tau<\infty$.

This finishes the proof of *Lemma* 3.2. □

We can continue now with the proof of *Lemma* 3.1. Let

$$b=\min\{1-\varepsilon,\frac{1-\epsilon}{4\tilde{C}}\}$$

where $\varepsilon>0$ arbitrary and $\tilde{C}$ is given by Lemma 3.2. Consider the ball of radius b centered at 0

$$\mathcal{B}=B(0,b)$$

Using *Lemma* 3.2, part *(ii)* with $u_2=0$ we have, for $u_1\in\mathcal{B}$

$$\|Nu_1\|_Y \leq \tilde{C}(\|u_1\|_Y^2+\|u_1\|_Y)\|u_1\|_Y \leq \tilde{C}2b\cdot b<b$$

which means that the ball $\mathcal{B}$ is invariant under the action of the operator N.

Also, using again *Lemma* 3.2, part *(ii)*, we have, for $u_1, u_2 \in \mathcal{B}$

$$\begin{aligned}\|Nu_1 - Nu_2\|_Y &\leq \tilde{C}(\|u_1\|_Y^2 + \|u_2\|_Y^2 + \|u_1\|_Y + \|u_2\|_Y)\|u_1 - u_2\|_Y \\ &\leq \tilde{C}4b\|u_1 - u_2\|_Y \leq (1-\epsilon)\|u_1 - u_2\|_Y\end{aligned}$$

which shows that $N : \mathcal{B} \to \mathcal{B}$ is a contraction. This finishes the proof of *Lemma* 3.1 and of *Theorem* 3.1. □

4 Linear Estimates

Consider the linear Schrödinger equation with a potential in two space dimensions:

$$\begin{cases} i\frac{\partial u}{\partial t} = (-\Delta + V(x))u \\ u(0) = u_0. \end{cases}$$

It is known that if V satisfies hypothesis (H1)(i) and (ii) then the radiative part of the solution, i.e. its projection onto the continuous spectrum of $H = -\Delta + V$, satisfies the estimates:

$$\|e^{-iHt}P_c u_0\|_{L^2_{-\sigma}} \leq C_M \frac{1}{(1+|t|)\log^2(2+|t|)}\|u_0\|_{L^2_\sigma} \tag{23}$$

for some constant $C_M > 0$ independent of u_0 and $t \in \mathbb{R}$, see [16, Theorem 7.6], and

$$\|e^{-iHt}P_c u_0\|_{L^p} \leq \frac{C_p}{|t|^{1-2/p}}\|u_0\|_{L^{p'}} \tag{24}$$

for some constant $C_p > 0$ depending only on $p \geq 2$ and p' given by $p'^{-1} + p^{-1} = 1$. The case $p = \infty$ in (24) is proven in [22]. The conservation of the L^2 norm, see [5, Corollary 4.3.3], gives the $p = 2$ case:

$$\|e^{-iHt}P_c u_0\|_{L^2} = \|u_0\|_{L^2}.$$

The general result (24) follows from Riesz-Thorin interpolation.

We would like to extend this estimates to the linearized dynamics around the center manifold. In other words we consider the linear equation, with initial data at time s,

$$\begin{cases} i\frac{\partial z}{\partial t} &= (-\Delta + V(x))z + \gamma P_c[2|\psi_E(t)|^2 z + \psi_E^2(t)\bar{z} + Dh|_{a(t)}(\langle\psi_0, 2|\psi_E|^2 z + \psi_E^2\bar{z}\rangle)] \\ z(s) &= v. \end{cases}$$

Note that this is a nonautonomous problem as the bound state ψ_E around which we linearize may change with time.

By Duhamel's principle we have:

$$\begin{aligned} z(t) = e^{-iH(t-s)}P_c v(s) - i\int_s^t e^{-iH(t-\tau)}\gamma P_c[2|\psi_E|^2 z + \psi_E^2\bar{z} + \\ Dh|_{a(\tau)}(\langle\psi_0, 2|\psi_E|^2 z + \psi_E^2\bar{z}\rangle)]d\tau \end{aligned} \tag{25}$$

As in (14) we denote

$$\Omega(t,s)v \stackrel{def}{=} z(t). \tag{26}$$

In the next two theorems we will extend estimates of type (23)-(24) to the operator $\Omega(t,s)$ relying on the fact that $\psi_E(t)$ is small. It would be useful to find sufficient conditions under which our results generalize to large bound states. Such conditions have been obtained in one or three space dimensions, see [2, 13, 23, 7], unfortunately their techniques cannot be applied in the two space dimension case.

We start with estimates in weighted L^2 spaces:

Theorem 4.1 *There exists $\varepsilon_1 > 0$ such that if $\| <x>^\sigma \psi_E\|_{H^2} < \varepsilon_1$ then there exist constants $C,\ C_p > 0$ with the property that for any $t,\ s \in \mathbb{R}$ the following hold:*

$$\textit{(i)}\ \|\Omega(t,s)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \frac{C}{(1+|t-s|)\log^2(2+|t-s|)}$$

$$\textit{(ii)}\ \|\Omega(t,s)\|_{L^{p'} \to L^2_{-\sigma}} \leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}},\ \text{for any } \infty > p \geq 2 \text{ where } p'^{-1} + p^{-1} = 1$$

$$\textit{(iii)}\ \|\Omega(t,s)\|_{L^2_\sigma \to L^p} \leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}},\ \text{for any } p \geq 2$$

Before proving the theorem let us remark that *(i)* is a generalization of (23) while *(ii)* and *(iii)* are a mixture between (23) and (24). We have used all these estimates in the previous section. They are consequences of contraction principles applied to (25) and involve estimates for convolution operators based on (23) and (24). It will prove much more difficult to remove the weights from the estimates *(ii)* and *(iii)*, see Theorem 4.2.

Proof of Theorem 4.1

Fix $s \in \mathbb{R}$.

(i) By definition (see (26)), we have $\Omega(t,s)v = z(t)$ where $z(t)$ satisfies equation (25). We are going to prove the estimate by showing that the nonlinear equation (25) can be solved via contraction principle argument in an appropriate functional space. To this extent let us consider the functional space

$$X_1 := \{z \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^2)) | \sup_{t\in\mathbb{R}} (1+|t-s|)\log^2(2+|t-s|)\|z(t)\|_{L^2_{-\sigma}} < \infty\}$$

endowed with the norm

$$\|z\|_{X_1} := \sup_{t\in\mathbb{R}} \{(1+|t-s|)\log^2(2+|t-s|)\|z(t)\|_{L^2_{-\sigma}}\} < \infty$$

Note that the inhomogeneous term in (25):

$$z_0(t) \stackrel{\text{def}}{=} e^{-iH(t-s)}P_c v$$

satisfies $z_0 \in X_1$ and

$$\|z_0\|_{X_1} \leq C_M \|v\|_{L^2_\sigma} \tag{27}$$

because of (23).

We collect the z dependent part of the right hand side of (25) in a linear operator $L(s): X_1 \to X_1$,

$$[L(s)z](t) = -i \int_s^t e^{-iH(t-\tau)} \gamma P_c [2|\psi_E|^2 z + \psi_E^2 \bar{z} + Dh|_{a(\tau)}(\langle \psi_0, 2|\psi_E|^2 z + \psi_E^2 \bar{z}\rangle)] d\tau$$

In what follows we will show that L is a well defined bounded operator from X_1 to X_1 whose operator norm can be made less or equal to 1/2 by choosing ε_1 in the hypothesis sufficiently small. Consequently $Id - L$ is invertible and the solution of the equation (25) can be written as $z = (Id - L)^{-1} z_0$. In particular

$$\|z\|_{X_1} \leq (1 - \|L\|)^{-1} \|z_0\|_{X_1} \leq 2\|z_0\|_{X_1}$$

which, in combination with the definition of Ω, the definition of the norm in X_1 and estimate (27), finishes the proof of *(i)*.

It remains to prove that L is a well defined bounded operator from X_1 to X_1 whose operator norm can be made less than 1/2 by choosing ε_1 in the hypothesis sufficiently small. We have the following estimates:

$$\begin{aligned} \|L(s)z(t)\|_{L^2_{-\sigma}} &\leq \int_s^t \|e^{-iH(t-\tau)} P_c\|_{L^2_\sigma \to L^2_{-\sigma}} \cdot [3\| |\psi_E|^2(\tau) z(\tau)\|_{L^2_\sigma} \\ + \|Dh|_{a(\tau)}\|_{\mathbb{C}\to L^2_\sigma} &|\langle \psi_0 < x >^\sigma, 2|\psi_E|^2 < x >^{-\sigma} z(\tau) + \psi_E^2 < x >^{-\sigma} \bar{z}(\tau)\rangle| d\tau \\ &\leq \int_s^t \|e^{-iH(t-\tau)} P_c\|_{L^2_\sigma \to L^2_{-\sigma}} \cdot [3\| |\psi_E|^2(\tau) z(\tau)\|_{L^2_\sigma} \\ &\quad + \|Dh|_{a(\tau)}\|_{\mathbb{C}\to L^2_\sigma} \|\psi_0\|_{L^2_\sigma} 3\|\psi_E^2\|_{L^\infty} \|z(\tau)\|_{L^2_{-\sigma}}] \end{aligned}$$

On the other hand

$$\| |\psi_E|^2 z\|_{L^2_\sigma} \leq \|z\|_{L^2_{-\sigma}} \| < x >^{2\sigma} |\psi_E|^2\|_{L^\infty}, \text{ and } \| < x >^\sigma \psi_E\|^2_{L^\infty} \leq \varepsilon_1^2 \tag{28}$$

where the last inequality holds because of the Sobolev imbedding $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ and of the inequality

$$\| < x >^\sigma \psi_E\|_{H^2} \leq \varepsilon_1.$$

Also

$$\|Dh|_{a(\tau)}\| \leq \bar{C}\varepsilon_1, \text{as } |a(\tau)| < \delta.$$

Using the last three relations, as well as the estimate (23) and the fact that $z \in X_1$ we obtain that

$$\begin{aligned} \|L(s)\|_{X_1 \to X_1} &\leq \varepsilon_1 \sup_{t>0} [(1 + |t-s|) \log^2(2 + |t-s|) \times \\ \times \underbrace{\int_s^t \frac{1}{1 + |t-\tau| \log^2(2 + |t-\tau|)} \cdot \frac{1}{(1 + |\tau - s|) \log^2(2 + |\tau - s|)} d\tau}_{\mathcal{I}} &\leq 2C_1 \varepsilon_1 \end{aligned} \tag{29}$$

Indeed, in order to prove the above we will split $\mathcal{I}$ into $A+B$ where

$$A=\int_s^{\frac{t+s}{2}}\frac{1}{(1+|t-\tau|)\log^2(2+|t-\tau|)}\frac{1}{(1+|\tau-s|)\log^2(2+|\tau-s|)}d\tau$$

for which we have the bound

$$\begin{aligned}|A|\le\frac{1}{(1+|\frac{t-s}{2}|)\log^2(2+|\frac{t-s}{2}|)}|\int_s^{\frac{t+s}{2}}\frac{1}{(1+|\tau-s|)\log^2(2+|\tau-s|)}|\\ \le C_2\frac{1}{(1+|\frac{t-s}{2}|)\log^3(2+|\frac{t-s}{2}|)}\end{aligned}$$

Observing that $A=B$ and using the last estimate in (29) we obtain that

$$\|L\|_{X_1\to X_1}\le C_1\varepsilon_1\le 1/2$$

for ε_1 small enough.

(ii) By the definition of Ω it is sufficient to prove that the solution of (25) satisfies

$$\|z(t)\|_{L^2_{-\sigma}}\le\frac{C_p}{|t-s|^{1-\frac{2}{p}}}\|v\|_{L^{p'}},\ \text{for all } \infty>p\ge 2 \text{ where } p'^{-1}+p^{-1}=1. \tag{30}$$

We will use a similar functional analytic argument as in the proof of *(i)*. Fix $p, 2\le p<\infty$ and assume $v\in L^{p'},\ p'^{-1}+p^{-1}=1$. We will work in the following functional space:

$$X_2:=\{z\in C(\mathbb{R},L^2_{-\sigma}(\mathbb{R}^2)|\sup_{t\in\mathbb{R}}\|z(t)\|_{L^2_{-\sigma}}|t-s|^{1-\frac{2}{p}}<\infty\}$$

endowed with the norm

$$\|z\|_{X_2}:=\sup_{t\in\mathbb{R}}\|z(t)\|_{L^2_{-\sigma}}|t-s|^{1-\frac{2}{p}}<\infty.$$

Using the fact that $L^p\hookrightarrow L^2_{-\sigma}$ continuously and the estimate (24) we have $e^{-iH(t-s)}P_cv\in X_2$. In addition, for L defined in the proof of *(i)*, we have

$$\begin{aligned}\sup_{t>0}|t-s|^{1-\frac{2}{p}}\|L(s)z(t)\|_{L^2_{-\sigma}}\le\\ \sup_{t>0}|t-s|^{1-\frac{2}{p}}\int_s^t\|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma\to L^2_{-\sigma}}\cdot[\||\psi_E|^2(\tau)z(\tau)+\\ +\psi_E^2(\tau)\bar{z}(\tau)\|_{L^2_\sigma}+\|P_cDh|_{a(\tau)}\|_{\mathbb{C}\to L^2_\sigma}|\langle\psi_0,2|\psi_E|^2z(s)+\psi_E^2\bar{z}(s)\rangle|d\tau\\ \le\sup_{t>0}|t-s|^{1-\frac{2}{p}}\int_s^t\frac{(C+\|\psi_0\|_{L^2})\|\psi_E^2<x>^{2\sigma}\|_{L^\infty}}{(1+|t-\tau|)|\tau-s|^{1-\frac{2}{p}}\log^2(2+|t-\tau|)}<C_3\varepsilon_1^2.\end{aligned} \tag{31}$$

Using now the bounds (28) in (31), for ε_1 small enough, we obtain that the norm of $L(s)$ is less or equal to 1/2, i.e. the operator $Id-L(t,s)$ is invertible, which, as in the proof of *(i)*, finishes the proof of estimate *(ii)*.

(iii) We already know from part *(i)* that equation (25) has a unique solution in $L^2_{-\sigma}$ provided $v \in L^2_\sigma$. We are going to show that the right hand side of (25) is in L^p. Indeed

$$\|e^{-iH(t-s)}P_c v(s)\|_{L^p} \leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}}\|v(s)\|_{L^{p'}} \leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}}\|v(s)\|_{L^2_\sigma} \tag{32}$$

where the C_p's in the two inequalities are different, for the first inequality we used (24) while for the second we used the continuous embedding $L^2_\sigma \hookrightarrow L^{p'}, 1 \leq p' \leq 2$. For the remaining terms we combine (24) with $\|z\|_{X_1} < \infty$ obtained in part *(i)*:

$$\begin{aligned}
&\|i\int_s^t e^{-iH(t-\tau)}P_c(2|\psi_E^2 z(\tau) + \psi_E^2 \bar{z}(\tau))d\tau\|_{L^p} \\
&\leq \int_s^t \frac{2C_4}{|t-\tau|^{1-\frac{2}{p}}}\| < x >^\sigma \psi_E^2\|_\alpha\| < x >^{-\sigma} z(\tau)\|_{L^2} \\
&\leq \int_s^t \frac{2C_4}{|t-\tau|^{1-\frac{2}{p}}}\frac{C\varepsilon_1^2}{(1+|\tau-s|)\log^2(2+|\tau-s|)}d\tau \leq \frac{C}{|t-s|^{1-\frac{2}{p}}}
\end{aligned} \tag{33}$$

with $\frac{1}{\alpha} + \frac{1}{2} = \frac{1}{p'}$.

Similarly, we have

$$\begin{aligned}
&\|i\int_s^t e^{-iH(t-\tau)}\gamma P_c Dh|_{a(\tau)}\langle\psi_0, 2\|\psi_E|^2 z(s) + \psi_E^2\bar{z}(s)\rangle d\tau\|_{L^p} \\
&\leq \int_s^t \frac{C_5}{|t-\tau|^{1-\frac{2}{p}}}|\langle\psi_0, 2\|\psi_E|^2 z(s) + \psi_E^2\bar{z}(s)\rangle|d\tau \\
&\leq \int_s^t \frac{C_5}{|t-\tau|^{1-\frac{2}{p}}}|\|\psi_0\|_{L^2}\| < x >^\sigma \psi_E^2\|_{L^\infty}\| < x >^{-\sigma} z\|_{L^2} \\
&\leq C_6\int_s^t \frac{1}{|t-\tau|^{1-\frac{2}{p}}}\frac{1}{(1+|\tau-s|)(\log^2(2+|\tau-s|)}d\tau \leq \frac{C}{|t-s|^{1-\frac{2}{p}}}.
\end{aligned} \tag{34}$$

Plugging (32)-(34) into (25) we get:

$$\|z(t)\|_{L^p} \leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}}\|v\|_{L^2_\sigma}$$

which by the definition $\Omega(t,s) = z(t)$ finishes the proof of part *(iii)*. □

The next step is to obtain estimates for $\Omega(t,s)$ in unweighted L^p spaces. They are needed for controlling the cubic term in the operator N of the previous section.

Theorem 4.2 *Assume that* $\| < x >^\sigma \psi_E\|_{H^2} < \varepsilon_1$ *(where* ε_1 *is the one used in Theorem 4.1). Then for all* $t,\ s \in \mathbb{R}$ *the following estimates hold:*

$$(i)\ \|\Omega(t,s)\|_{L^1\cap L^{q'}\cap L^{p'}\to L^p} \leq \frac{C_{p,q'}\log(2+|t-s|)}{(1+|t-s|)^{1-\frac{2}{p}}},$$

for all $p,\ q',\ 2 \leq p < \infty,\ 1 < q' \leq 2,\ p'^{-1} + p^{-1} = 1;$

(ii) $\|\Omega(t,s)\|_{L^2\cap L^{q_0'}\to L^2}\leq C_{q_0'},$ for all $q_0',\ 1<q_0'<\frac{4}{3};$

(iii) for fixed $p_0>0$ and $1<q_0'<4/3$ and for any $2\leq p\leq p_0$

$$\|\Omega(t,s)\|_{L^{q'}\cap L^{p'}\cap L^{q_0'}\to L^p}\leq\frac{C_{p,q_0'}\log(2+|t-s|)^{\frac{1-2/p}{1-2/p_0}}}{|t-s|^{1-\frac{2}{p}}}$$

where

$$\frac{1}{q'}=\theta+\frac{1-\theta}{q_0'}$$

with

$$\theta=\frac{1-2/p}{1-2/p_0},\ \textit{i.e.}\ \frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{2}.$$

Note that *(iii)* is similar to the standard estimate for Schrödinger operators (24) except for the logarithmic correction and a smaller domain of definition. We will obtain it by interpolation from *(i)* and *(ii)*. The proof of *(i)* will rely on a fixed point technique for equation (36) while the proof of *(ii)* will rely on Strichartz inequalities.

It turns out that we need to regularize (25) in order to obtain *(i)* and *(ii)*. The inhomogeneous term has a nonintegrable singularity at $t=s$ when estimated in L^∞ :

$$\|e^{-iH(t-s)}P_cv\|_{L^\infty}\leq|t-s|^{-1}\|v\|_{L^1}.$$

Using estimates with integrable singularities at $t=s$, for example in $L^p,\ p<\infty$ see (24), would lead to a slower time decay in *(i)* and eventually will make it impossible to close the estimates for the operator N in the previous section. We avoid this by defining:

$$W(t)\stackrel{def}{=}[z(t)-e^{-iH(t-s)}P_c]v(s)\tag{35}$$

which, by plugging in (25), will satisfy the following "regularized" equation:

$$\begin{aligned}W(t)=&\underbrace{-i\int_s^t e^{-iH(t-\tau)}\gamma P_c[2|\psi_E(\tau)|^2e^{-iH(\tau-s)}P_cv(s)+\psi_E^2(\tau)e^{iH(\tau-s)}P_c\bar{v}(s)]d\tau}_{f(t)}\\&\underbrace{-i\int_s^t e^{-iH(t-\tau)}\gamma P_cDh|_{a(\tau)}\langle\psi_0,2|\psi_E|^2e^{-iH(\tau-s)}P_cv(s)+\psi_E^2e^{iH(\tau-s)}P_c\bar{v}(s)\rangle d\tau}_{\tilde{f}(t)}\\&\underbrace{-i\int_s^t e^{-iH(t-\tau)}P_c(2|\psi_E|^2W(\tau)+\psi_E^2\bar{W}(\tau))d\tau}_{g(t)}\\&\underbrace{-i\int_s^t e^{-iH(t-\tau)}\gamma P_cDh|_{a(\tau)}\langle\psi_0,2|\psi_E|^2W(s)+\psi_E^2\bar{W}(s)\rangle d\tau}_{\tilde{g}(t)}\end{aligned}\tag{36}$$

Some other new notations are necessary for the sake of easy reference. We will denote by $T(t,s)$ the operator which associates to the initial data at time s, v, the function $W(t)$, so that

$$T(t,s)v \stackrel{def}{=} W(t) \tag{37}$$

which will be related to the operator $\Omega(t,s) = z(t)$ (see (14)) by

$$\Omega(t,s) = T(t,s) + e^{-iH(t-s)}P_c. \tag{38}$$

For T we can not only extend the estimate in Theorem 4.1 *(ii)* to the case $p = \infty$ but also obtain a nonsingular version of it:

Lemma 4.1 *Assume that $\| < x >^\sigma \psi_E\|_{H^2} < \varepsilon_1$ (where ε_1 is the one used in Theorem 4.1). Then for each $1 < q' \le 2$ there exists the constant $C_{q'} > 0$, $C_{q'} \to \infty$ as $q' \to 1$, such that for all $t,\ s \in \mathbb{R}$ we have:*

$$\|T(t,s)\|_{L^1 \cap L^{q'} \to L^2_{-\sigma}} \le \frac{C_{q'}}{1+|t-s|}.$$

Proof of the Lemma: Fix q', $1 < q' \le 2$. Consider equation (36) with $s \in \mathbb{R}$ arbitrary and $v \in L^1 \cap L^{q'}$. We are going to show that (36) has a unique solution in $C(\mathbb{R}, L^2_{-\sigma})$ satisfying:

$$\|W(t)\|_{L^2_{-\sigma}} \le \frac{C_{q'}}{1+|t-s|} \max\{\|v\|_{L^1},\ \|v\|_{L^{q'}}\}$$

which will be equivalent to the conclusion of the Lemma via the definition of T (37).

Let us observe that it suffices to prove this estimate only for the forcing term $f(t) + \tilde{f}(t)$ because then we will be able to do the contraction principle in the functional space (in time and space) in which $f(t)+\tilde{f}(t)$ will be, and thus obtain the same decay for W as for $f(t)+\tilde{f}(t)$.

Indeed, this time we will consider the functional space

$$X_3 := \{u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^2) | \sup_{t\in\mathbb{R}} \|u(t)\|_{L^2_{-\sigma}}(1+|t-s|) < \infty\}$$

endowed with the norm

$$\|u\|_{X_3} := \sup_{t\in\mathbb{R}}\{\|u(t)\|_{L^2_{-\sigma}}(1+|t-s|)\} < \infty$$

We have

$$\begin{aligned}
&\sup_{t>0}(1+|t-s|)\|L(s)u(t)\|_{L^2_{-\sigma}} \le \\
&\sup_{t>0}(1+|t-s|)\int_s^t \|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma \to L^2_{-\sigma}} \times [2\| < x >^\sigma |\psi_E|^2(\tau) < x >^\sigma < x >^{-\sigma} u(\tau)\|_{L^2_\sigma} \\
&\qquad + \|Dh\|_{L^2_\sigma}\|\psi_0\|_{L^2}\| < x >^\sigma \psi_E^2\|_{L^\infty}\|u(\tau - s)\|_{L^2_{-\sigma}}]d\tau \\
&\le \sup_{t>0}(1+|t-s|)\int_s^t \frac{C_7\|\psi_E^2 < x >^{2\sigma}\|_{L^\infty}}{(1+|t-\tau|)(1+|\tau-s|)\log^2(2+|\tau-s|)} < C_8\varepsilon_1
\end{aligned} \tag{39}$$

Using the bounds (28) in (39) we obtain that for ε_1 small enough the norm of $L(t,s)$ in X is less then one, i.e. the operator $Id - L(t,s)$ is invertible.

We need now to estimate $f(t) + \tilde{f}(t)$:

$$||f(t) + \tilde{f}(t)||_{L^2_{-\sigma}} \leq \int_s^t \frac{2C_M|||\psi_E|^2 < x >^\sigma e^{-iH(\tau-s)} P_c v(s)||_{L^2}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} d\tau \tag{40}$$

(where we used the estimate (23))

We will split now (40) into two parts to be estimated differently:

$$||f + \tilde{f}||_{L^2_{-\sigma}} \leq \underbrace{\int_s^{s+1} \dots}_{\mathcal{I}} + \underbrace{\int_{s+1}^t \dots}_{\mathcal{II}} \tag{41}$$

Then, we have:

$$|\mathcal{I}| \leq \int_s^{s+1} \frac{2C_M|||\psi_E|^2 < x >^\sigma e^{-iH(\tau-s)} P_c v(s)||_{L^2}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} \leq$$

$$\leq \frac{2C_M}{(1+|t-s-1|)\log^2(2+|t-s-1|)} \int_s^{s+1} ||e^{-iH(\tau-s)} P_c v(s)||_{L^q} \cdot \underbrace{|||\psi_E|^2 < x >^\sigma ||_{L^\alpha}}_{\leq \text{fixed constant}} \leq$$

$$\leq \frac{C_9}{(1+|t-s-1|)\log^2(2+|t-s-1|)} \int_s^{s+1} ||v(s)||_{L^{q'}} (\frac{1}{\tau - s})^{1-\frac{2}{q}} d\tau$$

$$\leq \frac{C_{10}||v(s)||_{L^{q'}}}{(1+|t-s-1|)\log^2(2+|t-s-1|)} \leq C_{11} \frac{1}{1+|t-s|} \|v(s)\|_{L^{q'}}$$

with $\frac{1}{\alpha} + \frac{1}{q} = \frac{1}{2}$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

For the second integral we have:

$$|\mathcal{II}| \leq \int_{s+1}^t \frac{2C_M|||\psi_E|^2 < x >^\sigma ||_{L^2} ||e^{-iH(\tau-s)} P_c v(s)||_{L^\infty}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} d\tau \leq$$

$$\leq \int_{s+1}^t \frac{2C_M|||\psi_E|^2 < x >^\sigma ||_{L^2}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} \cdot \frac{1}{|\tau - s|} ||v(s)||_{L^1} d\tau$$

$$\leq \frac{C_{12}}{1+|t-s|} ||v(s)||_{L^1}$$

Let us observe that the last two estimates are for the case when $t > s+1$. If $s < t < s+1$ we have

$$||f + \tilde{f}||_{L^2_{-\sigma}} \leq \int_s^t \frac{2C_M|||\psi_E|^2 < x >^\sigma e^{-iH(\tau-s)} P_c v(s)||_{L^2}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} d\tau$$

$$\leq 2C_M \int_s^t \frac{\|\psi_E^2 < x >^\sigma \|_{L^\alpha} \|e^{-iH(\tau-s)} P_c v(s)\|_{L^q}}{(1+|t-\tau|)(\log^2(2+|t-\tau|))} d\tau$$

$$\leq C_{13} \int_s^t (\frac{1}{\tau - s})^{1-\frac{2}{q}} d\tau \|v(s)\|_{q'} \leq C\|v(s)\|_{L^{q'}}$$

with $\frac{1}{\alpha} + \frac{1}{q} = \frac{1}{2}$.

Combining the last three estimates we get the lemma. □

We can now proceed with the proof of Theorem 4.2.

Proof of Theorem 4.2:

(i) Because of estimate (24) and relation (38) it suffices to prove *(i)* for $T(t,s)$.

Consider equation (36) with arbitrary $s \in \mathbb{R}$ and $v \in L^1 \cap L^{q'}$. In the previous Lemma we showed that the solution $W(t) \in L^2_{-\sigma}$. Now we show that it is actually in L^p for all $2 \le p < \infty$. Fix such a p. Then:

$$||W(t)||_{L^p} \le ||f(t) + \tilde{f}(t)||_{L^p} + \int_s^t ||e^{-iH(t-\tau)}P_c||_{L^{p'} \to L^p}(2 + \|Dh|_{a(\tau)}\|_{L^p}\|\psi_0\|_{L^p})$$
$$|||\psi_E|^2 < x >^{\sigma} < x >^{-\sigma} |W(\tau)|||_{L^{p'}} d\tau$$
$$\le ||f(t) + \tilde{f}(t)||_{L^p} + \int_s^t \frac{2C_{14}}{(t-\tau)^{1-\frac{2}{p}}} |||\psi_E|^2 < x >^{\sigma} ||_{L^\alpha}|| < x >^{-\sigma} W(\tau)||_{L^2} \tag{42}$$

(with $\frac{1}{\alpha} + \frac{1}{2} = \frac{1}{p'}$)

The estimate for $f(t) + \tilde{f}(t)$ is similar , but this time the term $|||\psi_E|^2 e^{-iH(\tau-s)}P_c v||_{L^{p'}}$ is controlled for $s + 1 < \tau < t$ by

$$||\psi_E^2||_{L^{p'}}||e^{-iH(\tau-s)}P_c v||_\infty \le \frac{C_{15}}{|t-s|}||v||_{L^1}$$

and for $s \le \tau < s + 1$ by

$$\|\psi_E\|_{L^\alpha}\|e^{-iH(\tau-s)}P_c v\|_{L^q} \le \frac{C_{16}}{(\tau - s)^{1-\frac{2}{q}}}$$

where $\alpha^{-1} + q^{-1} = p'^{-1}$ and $q^{-1} + q'^{-1} = 1$.

Using now the previous Lemma to estimate the term $|| < x >^{-\sigma} W(\tau)||_{L^2}$ and replacing in (42) we get:

$$||W(t)||_{L^p} \le \frac{C_{17}\log(1 + |t-s|)}{(1 + |t-s|)^{1-\frac{2}{p}}} \max\{||v||_{L^1}, ||v||_{L^{q'}}\}$$

with $1 < q' \le 2$ which is equivalent to

$$\|T(t,s)\|_{L^1 \cap L^{q'} \to L^p} \le \frac{C_{17}\log(1 + |t-s|)}{(1 + |t-s|)^{1-\frac{2}{p}}} \tag{43}$$

for all $2 \le p < \infty$ and $1 < q' \le 2$. This finishes the proof for part *(i)*.

(ii) Recalling the equation for W (36), let us observe that we have

$$\| \int_s^t e^{-iH(t-\tau)}P_c(|\psi_E|^2 W(\tau) + \psi_E^2 \bar{W}(\tau))d\tau\|_{L^2} \le C_S(\int_s^t \||\psi_E|^2 W(\tau)\|_{L^{\rho'}}^{\gamma'})^{\frac{1}{\gamma'}}$$
$$\le C_S(\int_s^t \|\psi_E^2 < x >^{\sigma} \|_{L^{\frac{2\gamma'}{2-\gamma'}}}^{\gamma'} \|W(\tau) < x >^{-\sigma} \|_{L^2}^{\gamma'} d\tau)^{\frac{1}{\gamma'}} \le C_S \varepsilon_1^2 \int_s^t \frac{1}{(1 + |\tau - s|)}^{\gamma'(1-\frac{2}{q_0})} d\tau$$
$$\le C_{18}\|v\|_{q_0'} \tag{44}$$

where for the first inequality we used the Strichartz estimate

$$(\mathcal{T}f)(t) = \int_s^t e^{-iH(t-\tau)} f(\tau) d\tau : L^{\gamma'}(0,T;L^{\rho'}) \to L^\infty(0,T;L^2)$$

with (γ, ρ) with $\gamma \geq 2$. For the second inequality we used Hölder's inequality and for the third one we used (30) combined with (35) and (24). Finally the last inequality holds when $\gamma'(1-\frac{2}{q_0}) > 1$ which happens for $q_0 > 2\gamma \geq 4$.

Also, we have the estimates

$$\begin{aligned}
&\| \int_s^t e^{-iH(t-\tau)} \gamma P_c Dh|_{a(\tau)} \langle \psi_0, 2|\psi_E|^2 W(\tau) + \psi_E^2 \bar{W}(\tau)\rangle d\tau \|_{L^2} \\
&\leq C_S(\int_s^t \|Dh|_{a(\tau)} \langle \psi_0, 2|\psi_E|^2 W(\tau) + \psi_E^2 \bar{W}(\tau)\rangle \|_{L^{\gamma'}}^{\rho'} d\tau)^{\frac{1}{\gamma'}} \\
&\leq \varepsilon_1 C_{19}(\int_s^t |\langle \psi_0, 2|\psi_E|^2 W(\tau) + \psi_E^2 \bar{W}(\tau)\rangle|^{\gamma'})^{\frac{1}{\gamma'}} \\
&\leq \varepsilon_1 C_{20}(\int_s^t (\|\psi_0\|_{L^2} \|\psi_E^2 < x >^\sigma \|_{L^\infty} \|W(\tau)\|_{L^2_{-\sigma}})^{\gamma'} d\tau)^{\frac{1}{\gamma'}} \\
&\leq \varepsilon_1 C_{21}(\int_s^t \frac{1}{(1+|\tau - s|)^{(1-\frac{2}{q_0})\gamma'}} d\tau)^{\frac{1}{\gamma'}} \|v(s)\|_{L^{q_0'}} \leq \varepsilon_1 C_{22} \|v\|_{L^{q_0'}} \qquad (45)
\end{aligned}$$

where for the first inequality we used Strichartz estimate as before and for the second inequality we use the fact that $Dh|_{a(\tau)}$ is bounded in H^2 and thus in any L^p and its norm is small. For the fourth inequality we used the fact that $\|\psi_0\|_{L^2}$ and $\||\psi_E|^2 < x >^\sigma \|_{L^\infty}$ are bounded and small. Finally the last inequality holds, as before, for $q_0 > 2\gamma \geq 4$.

For $f(t) + \tilde{f}(t)$ we'll need to estimate differently the short time behavior and the long time behavior, namely:

$$f(t) + \tilde{f}(t) = \underbrace{\int_s^{s+1} \dots}_{\mathcal{I}} + \underbrace{\int_{s+1}^t \dots}_{\mathcal{II}}$$

We have:

$$\begin{aligned}
|\mathcal{I}|_{L^2} &\leq 2\| \int_s^{s+1} e^{-iH(t-\tau)} \gamma P_c [|\psi_E|^2 e^{-iH(\tau-s)} P_c v(s) + Dh(\langle \psi_0, 2|\psi_E|^2 e^{-iH(\tau-s)} v(s)\rangle)] \|_{L^2} \\
&\leq C_{23} \int_s^{s+1} |||\psi_E|^2 e^{-iH(\tau-s)} P_c v(s)||_{L^2} + |\langle \psi_0, 2|\psi_E|^2 e^{-iH(\tau-s)} v(s)\rangle| d\tau \\
&\leq C_{24} \int_s^{s+1} |||\psi_E|^2||_{L^\alpha} ||e^{-iH(\tau-s)} P_c v(s)||_{L^{q_0}} + \|\psi_0\|_{L^2} \||\psi_E|^2\|_{L^\alpha} \|e^{-iH(\tau-s)} v(s)\|_{L^{q_0}} d\tau \\
&\leq C_{25} \int_s^{s+1} \frac{1}{(\tau - s)^{1-\frac{2}{q_0}}} d\tau ||v||_{L^{q_0'}} \leq C_{26} ||v||_{L^{q_0'}}
\end{aligned}$$

(where we used the fact that the operator e^{-iHt} preserves the L^2 norm, and $\frac{1}{\alpha} + \frac{1}{q_0} = \frac{1}{2}$). One estimates similarly the terms containing ψ_E^2 instead of $|\psi_E|^2$.

We continue by estimating $\mathcal{II}$:

$$
\begin{aligned}
|\mathcal{II}|_{L^2} &\leq \| \int_{s+1}^{t} |\psi_E|^2 e^{-iH(\tau-s)} P_c v d\tau \|_{L^2} \\
&\leq (\int_{s+1}^{t} \| |\psi_E|^2 e^{-iH(\tau-s)} P_c v \|_{q_0'}^{\alpha'} d\tau)^{\frac{1}{\alpha'}} \\
&+ (\int_{s}^{t} \|\psi_0\|_{L^2} \| |\psi_E|^2 \|_{L^\alpha} \| e^{-iH(\tau-s)} v \|_{L^{q_0}}^{\alpha'} d\tau)^{\frac{1}{\alpha'}} \\
\leq C_{27} (\int_{s+1}^{t} \| e^{-iH(\tau-s)} P_c v \|_{q_0}^{\alpha'})^{\frac{1}{\alpha'}} &\leq C_{28} (\int_{s+1}^{t} \frac{1}{(\tau-s)^{\alpha'(1-\frac{2}{q_0})}} d\tau)^{\frac{1}{\alpha'}} \|v\|_{q_0'} \leq C_{29} \|v\|_{q_0'}
\end{aligned}
$$

where for the first inequality we used the fact that the L^2 norm is preserved by the operator $e^{-iHt}P_c$. For the second inequality we used the Strichartz estimate

$$
(\mathcal{T} f)(t) = \int_{s}^{t} e^{-iH(t-\tau)} f(\tau) d\tau : L^{q_0'}(0,T; L^{\alpha'}) \to L^{\infty}(0,T; L^2)
$$

for the $f(t)$ term. For the $\tilde{f}(t)$ term we used similarly the same Strichartz estimate, the fact that $\|Dh\|_{L^{q_0'}}$ is bounded (as it is in any L^p norm), and we estimated the scalar product by the product $\|\psi_0\|_{L^2} \| |\psi_E|^2 \|_{L^\alpha} \| e^{-iH(\tau-s)} v(s) \|_{L^{q_0}}$. For the third inequality we used Hölder's inequality and the fact that the $\| |\psi_E|^2 \|_{L^\beta}, \| |\psi_E|^2 \|_{L^\alpha}, \|\psi_0\|_{L^2} \leq C, \forall t$ (where $\frac{1}{q_0'} = \frac{1}{q_0} + \frac{1}{\beta}$ and $\frac{1}{2} = \frac{1}{\alpha} + \frac{1}{q_0}$). Finally the last inequality holds because $\alpha'(1-\frac{2}{q_0}) > 1$, as $q_0 > \frac{2}{\alpha}$.

Let us observe that we assumed that $t > s+1$. If $s < t < s+1$, only the estimate for $\mathcal{I}$ will suffice, where the upper limit of integration $s+1$ should be replaced by t.

Combining the estimates for $\mathcal{I}, \mathcal{II}$, (44) and (45) we have that $W(t)$ is uniformly bounded in L^2 which, by (37), implies

$$
\|T(t,s)\|_{L^{q_0'} \to L^2} \leq C_{q_0'}, \text{ for all } t, \ s \in \mathbb{R}. \tag{46}
$$

Using now (38) and (24) with $p = p' = 2$ we obtain *(ii)*. $\square$

(iii) We start from (43):

$$
\|T(t,s)\|_{L^1 \cap L^{q_0'} \to L^p} \leq \frac{C_{p,q_0'} \log(2+|t-s|)}{(1+|t-s|)^{1-\frac{2}{p}}}
$$

and (46):

$$
\|T(t,s)\|_{L^{q_0'} \to L^2} \leq C_{q_0'}, \ 1 < q_0' < \frac{4}{3}.
$$

We can now use the Riesz-Thorin interpolation between the spaces $L^1 \cap L^{q_0'}$ and $L^{q_0'}$ as starting spaces and between L^p and L^2 as arrival spaces to get the claimed estimate. Indeed, it suffices to take as in the statement $\theta = \frac{1-2/p}{1-2/p_0}$, and use it with the above two relations to get

$$
\|T(t,s)\|_{L^{q'} \cap L^{p'} \cap L^{q_0'} \to L^p} \leq \frac{C_{p,q_0'} \log(2+|t-s|)^{\frac{1-2/p}{1-2/p_0}}}{|t-s|^{1-\frac{2}{p}}}
$$

with $1 < q_0' < \frac{4}{3}, p \geq 2$ and

$$\begin{cases} \frac{1}{q'} = \theta + \frac{1-\theta}{q_0'} \\ \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{2}, \theta = \frac{1-2/p}{1-2/p_0} \end{cases}$$

Using now (38) and (24) we obtain the claimed estimate for $\Omega(t,s)$. $\square$

5 Conclusions.

We have established that the solution starting from small and localized initial data will approach, as $t \to \pm\infty$, the center manifold formed by the nonlinear bound states (solitary waves). However we have not been able to decide whether the solution will approach exactly one solitary wave as in the 1-d and 3-d case, see for example [4, 19]. Here is the main reason:

The long time dynamics on the center manifold is given by the equation (12). Since

$$a(\pm\infty) - a(0) = \lim_{t\to\pm\infty} \int_0^t \frac{da}{dt} dt,$$

the existence of an asymptotic limit at $t = \pm\infty$ is equivalent to the integrability of the right hand side of (12) on $[0, +\infty)$ respectively $(-\infty, 0]$. The terms containing r^2 and r^3 are absolutely integrable because they are dominated by $(1 + |t|)^{2(2/p-1)}$, respectively $(1 + |t|)^{3(2/p-1)}$, which are integrable on $\mathbb{R}$ for $p > 4$. However, the linear terms in r do not decay fast enough to be absolutely integrable. It is possible though that a combination of decay and oscillatory cancellations would render it integrable. We think that is only a matter of time until a suitable treatment of this term is found. Note that, in the 1-d and 3-d cases, the linear terms in r were absolutely integrable in time, see for example [4, 19]. But these estimates relied on the integrable decay in time of the Schrödinger operator in L^∞ norm in 3-d, respectively on the large power nonlinearity to compensate for the linear growth in time introduced by virial type estimates in 1-d. None would work for our cubic NLS in 2-d.

The situation is even more complex and possible more interesting when the center manifold has more than one branch (more than one connected component). For simplicity, consider the case when hypothesis (H1) part (iii) is relaxed to allow for two, simple, negative eigenvalues $E_0 < E_1$ with corresponding normalized eigenvectors $\psi_0,\ \psi_1$. In this case the center manifold has two branches $\psi_{E_j} = a_j\psi_j + h_j(a_j),\ j = 0, 1$, each bifurcating from one eigenvector as described in Section 2. The decomposition into the evolution on the center manifold and the one away from it will now be:

$$u(t,x) = \underbrace{\sum_{j=0}^{1} (a_j(t)\psi_j(x) + h_j(a_j(t)))}_{\psi_{\mathrm{CM}}(t)} + r_m(t,x)$$

The equation for $r_m(t)$ remains essentially the same as (13) in Section 3, with ψ_E replaced by ψ_{CM} and the differential of h replaced by the sum of the differentials of $h_j,\ j = 0, 1$. However, one has to add to the right hand side of (13) the projection onto the continuous spectrum of the interaction term between the branches:

$$2|\psi_{E_0}|^2\psi_{E_1} + \psi_{E_0}^2\bar{\psi}_{E_1} + 2\psi_{E_0}|\psi_{E_1}|^2 + \bar{\psi}_{E_0}\psi_{E_1}^2 \tag{47}$$

In principle one could use our techniques and obtain a decay in time for $r_m(t)$, hence collapse on the center manifold, provided one makes the ansatz that the term above, or at least its projection onto the continuous spectrum, decays in time. Such an ansatz needs to be supported by the analysis of the motion on the center manifold given now by a system of two ODE's, one for a_0 and one for a_1. Each of the equations will be similar to (12) but the projection of (47) onto ψ_0, respectively ψ_1, has to be added to the right hand side. Note that, in the 3-d case, under the additional assumption $2E_1 - E_0 > 0$, it has been shown that the evolution approaches asymptotically a ground state (a periodic solution on the branch bifurcating from ψ_0) except when the initial data is on a finite dimensional manifold near the excited state branch (the one bifurcating from ψ_1), see [27, 29, 30, 31]. But the authors' analysis relies heavily on the much better dispersive estimates for Schrödinger operators in 3-d compared to 2-d. The 2-d case remains open.

Returning now to the case of one branch center manifold in 2-d, an important question is whether its stability persists under time dependent perturbations. In [6] we showed that this is not the case in 3-d. The slower decay in time of the Schrödinger operator in 2-d compared to 3-d prevents us, yet again, from extending the technique in [6] to the 2-d setting.

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